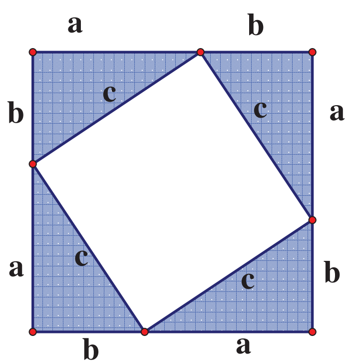
8.1 Pythagorean Theorem and Converse

The Pythagorean Theorem states that if a triangle is a right triangle, then the sum of the squares of the lengths of the legs equals the square of the hypotenuse lengths. That's a complicated way to say that if the legs of the triangle measure a and b and the hypotenuse measures c, then $a^2 + b^2 = c^2$. While you may have heard this in the past, we will now prove it.

Proof of the Pythagorean Theorem

There are many ways to prove the Pythagorean Theorem, but take a look at the following picture. We will refer to this for our proof.



In this picture we have a large square whose side lengths are equal to $a + b$ and an inner square whose side lengths are c. Notice that if we find the area of the large square and subtract the area of the triangles we get the area of the inner square. So let's do that algebraically.

The area of the larger square is:

$$(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

The area of each triangle is $\frac{1}{2}ab$ and since there are four of them, the total area of the triangles is $2ab$.

The area of the inner square is c^2.

This means the large square minus the triangles would look like this:

$$a^2 + 2ab + b^2 - 2ab = c^2$$

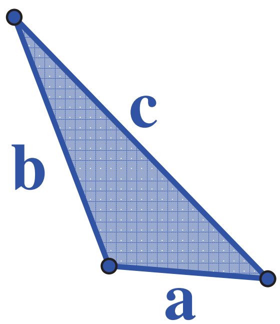
Notice that the $+2ab$ and the $-2ab$ cancel each other out (become zero), so we do get the result we expect which is that $a^2 + b^2 = c^2$.

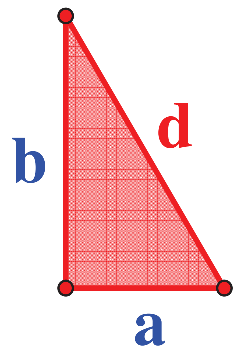
Do a search online to see if you can find another proof for this vital theorem.

Proof of the Pythagorean Theorem Converse

The converse of the Pythagorean Theorem states that if a triangle with side lengths a, b, and c has the property that $a^2 + b^2 = c^2$, then it is a right triangle. We will now prove this.

Assume you have a triangle with side lengths a, b, and c has the property that $a^2 + b^2 = c^2$. Now construct another triangle with side lengths a and b, but make it a right triangle this time with a hypotenuse of length d. The picture would look like this with the original triangle on the left (the one that we don't know whether it is a right triangle or not) and the new triangle on the right (the one we make specifically to be a right triangle).





Since we know the Pythagorean Theorem is true, we know that $a^2 + b^2 = d^2$ which means that $d = \sqrt{a^2 + b^2}$ by taking the square root of both sides.

This means that $d = c$ since $c = \sqrt{a^2 + b^2}$ as well by the original statement that for this triangle $a^2 + b^2 = c^2$. Since all three side lengths are the same, the two triangles are congruent which means that the first triangle must be a right triangle just like the second one we made.

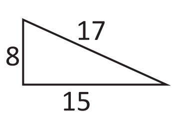
We can't use what has been called the LLP, or the Looks Like it Postulate. Just because the triangle on the left doesn't look like a right triangle, doesn't mean it actually isn't based on the facts we are given about it. The picture is inaccurate in this case.

Implications for the Pythagorean Theorem and its Converse

Now that we know both if a triangle is right then $a^2 + b^2 = c^2$ and if $a^2 + b^2 = c^2$ then the triangle is right, we can solve multiple types of problems. Given any two side lengths of a right triangle we can solve for the third side length using the Pythagorean Theorem. Given three side lengths of a triangle we can test if it's a right triangle using the Pythagorean Theorem converse.

Is it Right?

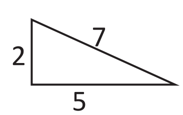
Because of the Pythagorean Converse, we can check whether a triangle is a right triangle or not. Consider the following two triangles. If their side lengths make the Pythagorean Theorem true, they are right.



$8^2 + 15^2 = 17^2$

$64 + 225 = 289$

True, so this is a right triangle.



$2^2 + 5^2 = 7^2$

$4 + 25 \neq 49$

False, $4 + 25$ is not 49, so it is not a right triangle.

Lesson 8.1

Answer the following questions.

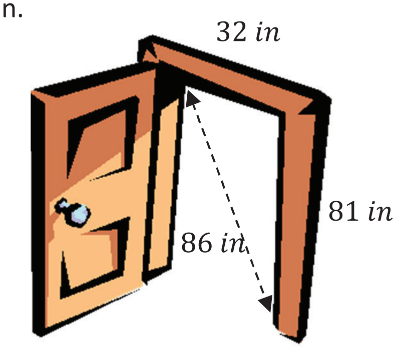
1. What is the Pythagorean Theorem in your own words?

2. What does the Pythagorean Theorem allow us to do?

3. What is the Pythagorean Theorem Converse in your own words?

4. What does the Pythagorean Theorem Converse allow us to do?

5. The door to your bathroom has never closed well. In fact, every time you try to use the bathroom, the cats bust open the door because it simply won't latch. You look at the door and it appears that the door frame is slightly tilted. The person who built your house claims that can't be true because he measured your door frame and found it be an exact right angle. He claims what you're seeing is an optical illusion.



  a. Without having a protractor, what could you do to see if he is correct without having a protractor?

  b. If you knew the door frame measurements were as pictured to the right, did the builder install your door frame correctly at a right angle?

6. Bob is building a triangular garden and needs fencing around it to keep the rabbits out. He has one section of fence measuring 40 ft, another measuring 42 ft, and a third measuring 58 ft. Bob says that after the fence is complete it will make a right triangle using the following argument: "First, I'll set-up the longest section of fence. Next, I'll attach the other two sections to either end of the long one. Finally, I'll swing the two shorter sections together. Since they must meet together, that makes it a right triangle."

  a. Is Bob correct that the garden fence will make a right triangle?

  b. If so, is Bob's argument correct for why it will make a right triangle?

  c. What would be a better argument?

Determine if the following triangles are right triangles or not using the Pythagorean Theorem Converse.

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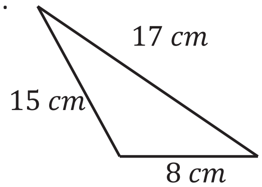

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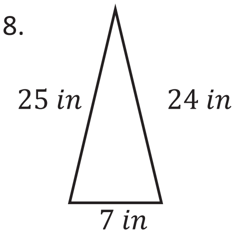

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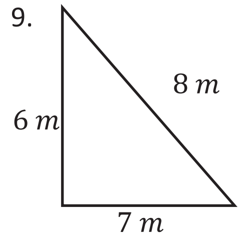

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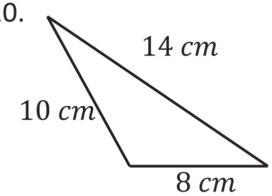

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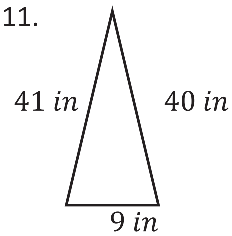

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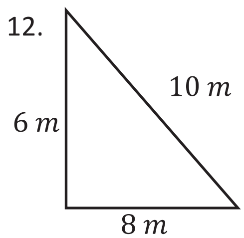

13. $a = 12\ ft$
$b = 16\ ft$
$c = 25\ ft$

14. $a = 12\ km$
$b = 35\ km$
$c = 37\ km$

15. $a = 10\ mm$
$b = 24\ mm$
$c = 27\ mm$

16. $a = 20\ ft$
$b = 21\ ft$
$c = 29\ ft$

17. $a = 5\ km$
$b = 12\ km$
$c = 17\ km$

18. $a = 5\ mm$
$b = 12\ mm$
$c = 13\ mm$